Generalization of the Kolmogorov $-5/3$ law of turbulence

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When the energy transfer function Π is a power-law function of the wave number k, $\Pi \propto k^{\alpha}$, the spectral dynamic equation of isotropic turbulence has a power-law solution, and the energy spectrum is $E(k)=C(\alpha)\Pi^{2/3}k^{-5/3}$, which is a generalization of the Kolmogorov $-5/3$ law $E(k)=K_0\epsilon^{2/3}k^{-5/3}$. The Kolmogorov law corresponds to the special case of $\alpha=0$ and $\Pi=\epsilon$. Here $C(\alpha)$ is a dimensionless coefficient and depends upon the exponent α, $K_0=C(0)$ is the Kolmogorov constant, and ϵ is the energy dissipation rate. $C(\alpha)$ is evaluated by numerically solving the spectral dynamic equation.

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The energy spectrum $E(k)$ of an isotropic turbulence satisfies the following spectral form of the von Kármán–Howarth equation [1]:

$$dE(k)/dt+2\nu k^2E(k)=T(k) .$$

Here $dE(k)/dt$ denotes the derivative of $E(k)$ with respect to time t, k is the wave number, ν is the kinematic viscosity, and $T(k)$ is the energy transfer spectrum function of the triad interaction between modes of turbulence. The energy transfer function $\Pi=\Pi(k)=\int_k^\infty T(k')dk'$ is the net rate of energy transfer from the wave numbers below k to those above. If Π is a constant independent of k and equals the energy dissipation rate ϵ, we have

$$E(k)=K_0\epsilon^{2/3}k^{-5/3} , \quad (1)$$

which is the celebrated Kolmogorov $-5/3$ law, K_0 being the Kolmogorov constant. The Kolmogorov $-5/3$ law (1) is a consequence of scaling properties of the hydrodynamic equation [2], and can be derived from the Navier-Stokes equation by statistical closure methods. The scaling properties imply that $E(k)\propto k^{\gamma}$ is a possible statistical solution of the Navier-Stokes equation even if γ takes a value other than $-\frac{5}{3}$. This possibility is explored here. $\Pi(k)=\text{const}$, a premise of the Kolmogorov $-5/3$ law, is a special case of the following more general expression of Π:

$$\Pi=\Pi(k)=\Pi_1k^{\alpha} . \quad (2)$$

Here Π_1 is a constant independent of k. This paper proves that if the energy transfer function Π is given by (2), then the energy spectrum

$$E(k)=C(\alpha)\Pi^{2/3}k^{-5/3} , \quad (3)$$

so $E(k)\propto k^{-5/3+2\alpha/3}$. Here $C(\alpha)$ is a dimensionless coefficient and depends upon the exponent α. Equation (3) is called the generalized $-5/3$ law and becomes the Kolmogorov law (1) when $\alpha=0$, $\Pi=\Pi_1=\epsilon$, and $C(0)=K_0$. Readers who are not interested in the derivation of (3) can skip the next two mathematical paragraphs.

Various statistical closure methods of the Navier-Stokes equation [3–5] yield the following expression for the energy transfer function Π:

$$\Pi=\int_k^\infty dk'\int_0^k dp\int_{\max(p,k'-p)}^{p+k'} dr\, S(k'|p,r) , \quad (4a)$$

$$S(k|p,r)=16\pi^2k^3pr[b(k,p,r)d(r,p,k)+b(k,r,p)d(p,r,k)]/\eta(k,p,r) , \quad (4b)$$

$$d(r,p,k)=q(r)[q(p)-q(k)] , \quad q(k)=E(k)/4\pi k^2 , \quad (4c)$$

$$\eta(k,p,r)=\eta(k)+\eta(p)+\eta(r) . \quad (4d)$$

Here $\eta(k)$ is the dynamic damping coefficient and $b(k,p,r)$ is a geometrical factor. Different closure methods give $\eta(k)$ quite different meanings. If we treated $\eta(k)$ as an optimal control parameter to minimize the approximation which is made during the derivation of (4), we would obtain

$$q(k)\eta(k)=2\pi\int_0^\infty dp\int_{\max(p,k-p)}^{p+k} dr\, kpr\{f(k,p,r)+f(k,r,p)+b^*(k,p,r)d(k,r,p)[\eta(p)-\eta(r)]\}/\eta(k,p,r)^2 , \quad (5a)$$

$$f(k,p,r)=b(k,p,r)d(r,k,p)[2\eta(p)+\eta(r)] , \quad (5b)$$

which is called the $\eta(k)$ equation [5]. The geometrical factors $b(k,p,r)$ and $b^*(k,p,r)$ are defined as follows:

$$b(k,p,r)=(p/k)(xy+z^3) , \quad b^*(k,p,r)=(pr/k^2)(yz+x^3) ,$$

where x, y, and z are the cosines of three angles of the triangle with sides k, p, and r.

When the energy transfer function Π is given by (2), Eqs. (4) and (5) have a power-law solution,

$$q(k)=Ak^m , \quad \eta(k)=Bk^n . \quad (6)$$

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Substituting (6) into (4) and (5), we obtain a set of algebraic equations for determining A, m, B, and n. Let $k'=k/u$, $p=vk'$, and $r=wk'$; after some manipulation, from (2), (4), and (6) we obtain

$$8+2m-n=\alpha\ , \tag{7}$$

$$\Pi_1=16\pi^2[A^2/B]I_1(m,n,\alpha)\ , \tag{8a}$$

$$I_1(m,n,\alpha)=\int_0^1 dv(v^{1-\alpha}-v)/\alpha\int_{\max(v,1-v)}^{1+v} dw\, w[b(1,v,w)w^m(v^m-1)+b(1,w,v)v^m(w^m-1)]/[1+v^n+w^n]\ . \tag{8b}$$

Similarly, if we let $p=vk$ and $r=wk$, from (5) and (6) we obtain

$$5+m-2n=0\ , \tag{9}$$

$$B^2/A=2\pi I_2(m,n)\ , \tag{10a}$$

$$I_2(m,n)=\int_0^\infty dv\, v\int_{\max(v,1-v)}^{1+v} dw\, w[b(1,v,w)w^m(1-v^m)(2v^n+w^n)+b(1,w,v)v^m(1-w^m)(2w^n+v^n)+b^*(1,v,w)(w^m-v^m)(v^n-w^n)]/[1+v^n+w^n]^2\ . \tag{10b}$$

From (7) and (9) we have

$$m=-\tfrac{11}{3}+2\alpha/3\ ,\quad n=\tfrac{2}{3}+\alpha/3\ . \tag{11}$$

Instead of A and B, we use the dimensionless coefficients

$$C=4\pi A/\Pi_1^{2/3}\ ,\quad D=B/\Pi_1^{1/3}\ . \tag{12}$$

Then, Eqs. (8a) and (10a) become

$$D/C^2=I_1(-\tfrac{11}{3}+2\alpha/3,\tfrac{2}{3}+\alpha/3,\alpha)\ , \tag{13a}$$

$$D^2/C=I_2(-\tfrac{11}{3}+2\alpha/3,\tfrac{2}{3}+\alpha/3)/2\ . \tag{13b}$$

When α is given, $I_1(-\frac{11}{3}+2\alpha/3,\ \frac{2}{3}+\alpha/3,\ \alpha)$ and $I_2(-\frac{11}{3}+2\alpha/3,\ \frac{2}{3}+\alpha/3)$ can be evaluated by using (8b) and (10b). Then, from (13) we determine the dimensionless coefficients C and D. Obviously C is a function of α, i.e., $C=C(\alpha)$. The change of the ratio $C(\alpha)/K_0$ with α is given in Fig. 1, and $K_0=C(0)$. From Eqs. (2), (6), (11), and (12), we have

$$E(k)=4\pi k^2q(k)=4\pi Ak^{m+2}=C[\Pi_1k^\alpha]^{2/3}k^{-5/3}=C\Pi^{2/3}k^{-5/3}\ ,$$

which is exactly the generalized $-5/3$ law (3).

In summary, when the energy transfer function Π is given by (2), the spectral dynamic equations have a power-law solution, and the energy spectrum $E(k)$ is expressed by (3), which is a generalization of the Kolmogorov $-5/3$ law. The Kolmogorov law (1) is just a special case of (3), corresponding to $\alpha=0$, $\Pi=\epsilon$, and $K_0=C(0)$. Figure 1 shows the change of $C(\alpha)/K_0$ with α.

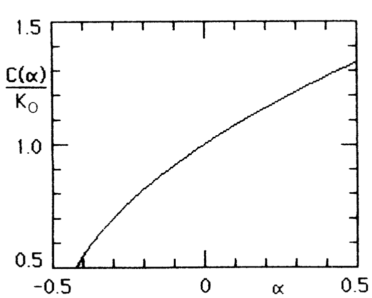

FIG. 1. $C(\alpha)/K_0$ vs α.

The above derivation of the generalized $-5/3$ law (3) is based upon a particular closure method. Actually, (3) can also be derived by other closure methods capable of deriving the Kolmogorov $-5/3$ law from the Navier-Stokes equation. Various closure methods [3–5] yield the same expression for the energy transfer function Π, but have different forms of the η equation (or response equations). If a closure method is capable of deriving the Kolmogorov $-5/3$ law, its closed spectral dynamic equations have a power-law solution (6). Then, its energy equation leads to (7) and its η (or response) equation leads to (9); hence we obtain the generalized $-5/3$ law (3). Of course, different closure methods have different forms of the function $I_2(m,n)$ corresponding to different forms of the η (or response) equation. Consequently, this might predict different values of the dimensionless coefficient $C(\alpha)$. The issue over which closure method is better has no effect upon the validity of the generalized $-5/3$ law (3) and will not be discussed here.

One might ask why and where the exponent α of (2) is different from zero. Before discussing the problem, it should be noticed that the energy transfer function Π approaches zero as $k\to 0$ or $k\to\infty$; hence, it is not realistic to model $\Pi=\Pi(k)$ by a power function of k for the whole wave number range of a real turbulent flow. However, it is possible to use (2) as a simple model of Π in some limited range. For example, the Kolmogorov model $\Pi=\text{const}=\epsilon(\alpha=0)$ is routinely used over a limited inertial range of real turbulent flows. In the inertial range, the viscous term $2\nu k^2E(k)$ is neglected, the spectral dynamical equation becomes $dE(k)/dt=T(k)$, and

$$\Pi=\int_k^\infty T(k')dk'=\int_k^\infty[dE(k')/dt]dk'+\text{const}\quad (k \text{ in inertial range})\ . \tag{14}$$

In an absolutely stationary turbulence $dE(k)/dt=0$, Π becomes a constant independent of k, the α of (2) is zero, and the Kolmogorov $-5/3$ law (1) is valid. When the macrostructure of a high Reynolds number turbulence is changing spatially and temporally, $dE(k)/dt\neq 0$, by (14) Π is no longer a constant in the inertial range. In this case, as pointed out by Lumley [6], the energy flux at the low wave number end of the inertial-range spectral pipeline is not equal to the energy flux at the high wave number end. When the macrostructure is changing spatially and temporally, the physical fact of Π changing with k in the inertial range should be taken into account in turbulence modeling [6]. In the inertial range of a decaying homogeneous turbulence, $dE(k)/dt$ is negative and by (14) Π increases with k; hence (2) with a positive α is a better model than Π=const. In the inertial range of a developing turbulence, initially only large scales are excited, then smaller and smaller scales are excited step by step, $dE(k)/dt$ is positive, by (14) Π decreases with k, and hence (2) with a negative α is a better model than Π=const. Although the macrostructure of a turbulence is changing, $E(k)/[dE(k)/dt]$ is still greater than the relaxation time of modal triad interactions in the inertial range, so the spectral dynamic equations (4) and (5) are still valid. The quasistationarity condition implies that $|\alpha|$ is not large. When $|\alpha|<0.5$, the deviation of the exponent $-5/3+2\alpha/3$ of the energy spectrum (3) from the exponent $-5/3$ of (1) is less than 20%, which is not easy to be observed over a quite limited wave-number range. However, according to Fig. 1, the corresponding relative change in $C(\alpha)$ may be higher than 30%, which might be one of the reasons why the experimental values of the Kolmogorov constant are widely scattered.

Of course, $dE(k)/dt\neq0$ is not the unique physical situation where (2) is a better model than Π=const. In numerical simulations or some experiments [7], a quite narrow inertial range is observed around 0.1 k_d. Here k_d is the Kolmogorov wave number. In the narrow range around 0.1 k_d, the viscous term $2\nu k^2E(k)$ is positive, $\Pi=2\nu\int_k^\infty k'^2E(k')dk'$ decreases with k even if we assume $dE(k)/dt=0$, and (2) with a negative α is a better model than Π=const. In short, the Kolmogorov model $\Pi=\text{const}=\epsilon$ is the simplest inertial-range model, and is a special case ($\alpha=0$) of (2), which represents an improvement over Π=const to take into account the long-range effect of the viscous term as well as the nonstationarity effect. We should notice that the exponent α of (2) is not a universal constant; its value depends upon the wave-number range to be studied and how the macrostructure of a turbulent flow is changing, spatially and temporally.

It is well known [1] that the intermittency of turbulence may lead to inertial-range energy spectrum to deviate from the Kolmogorov $-5/3$ law (1). The main purpose of this paper is to prove the generalized $-5/3$ law (3). It is not intended to give any contribution to the intermittency problem. According to (3), the deviation of the energy spectrum from the Kolmogorov $-5/3$ law might be a result of the wave-number dependence of the energy transfer function Π, in addition to the intermittency effect. The Kolmogorov $-5/3$ law (1) is a consequence of two premises, the premise of self-similarity of small scales (or the premise of global scaling [2]) and the premise of constant Π [$\Pi(k)=\text{const}=\epsilon$]. The self-similarity premise implies $E(k)\propto k^\gamma$ while the premise of constant Π specifies $\gamma=-5/3$ and selects the Kolmogorov $-5/3$ law. One interesting problem is the question over what will happen if the first premise (self-similarity or global scaling) is not valid, which is the subject of various intermittency models such as the local multifractal scaling model [2]. Another interesting problem is the question over what will happen if the second premise (constant Π) is not valid, which is studied here, and the generalized $-5/3$ law (3) is derived.

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